

HOLOMORPHIC ALMOST MODULAR FORMS

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ABSTRACT. Holomorphic almost modular forms are holomorphic functions of the complex upper half plane which can be approximated arbitrarily well (in a suitable sense) by modular forms of congruence subgroups of large index in $\mathrm{SL}(2, \mathbb{Z})$. It is proved that such functions have a rotation-invariant limit distribution when the argument approaches the real axis. An example for a holomorphic almost modular form is the logarithm of $\prod_{n=1}^{\infty} (1 - \exp(2\pi i n^2 z))$. The paper is motivated by the author's studies [J. Marklof, Int. Math. Res. Not. **39** (2003) 2131-2151] on the connection between almost modular functions and the distribution of the sequence $n^2 x$ modulo one.

1. INTRODUCTION

Almost modular functions have recently been introduced in connection with the distribution of the sequence $n^2 x$ modulo one ($n = 1, 2, 3, \dots$). It is shown in [5] that, for every piecewise smooth periodic function $\psi : \mathbb{R} \rightarrow \mathbb{C}$ of period one, and for x uniformly distributed in $[0, 1)$, the error term

$$(1.1) \quad R_{\psi}^x(M) = \frac{1}{\sqrt{M}} \left(\sum_{n=1}^M \psi(n^2 x) - M \int_0^1 \psi(t) dt \right)$$

has a limit distribution as $M \rightarrow \infty$, which can be identified with the value distribution of a certain almost modular function. This observation resembles results by Heath-Brown [3] and Bleher [1], [2], who prove that error terms in lattice point problems for convex planar domains have limit distributions associated with *almost periodic functions* (in the sense of Besicovitch).

In the present work we draw attention to the holomorphic species of almost modular functions. A *holomorphic almost modular form* (HAMF) is defined as a holomorphic function of the complex upper half plane \mathfrak{H} to \mathbb{C} that can be approximated arbitrarily well (in a sense to be made precise in Sect. 3) by modular forms of congruence subgroups $\Gamma_1(N)$ in $\mathrm{SL}(2, \mathbb{Z})$, as $N \rightarrow \infty$. An example for a HAMF is the logarithm of

$$(1.2) \quad \prod_{n=1}^{\infty} (1 - e(n^2 z)),$$

where $e(z) := \exp(2\pi i z)$, cf. Sect. 3. As a consequence of the general limit theorem for almost modular functions [5, Th. 8.2], we will see in Sect. 4 that, for $\mathrm{Re} z$ uniformly distributed in $[0, 1)$,

$$(1.3) \quad (\mathrm{Im} z)^{1/4} \log \prod_{n=1}^{\infty} (1 - e(n^2 z))$$

has a rotation-invariant limit distribution in \mathbb{C} , as $\text{Im } z \rightarrow 0$. An explicit formula for the variance of the limit distribution is given in Sect. 5. The paper concludes with a short appendix (Section 6) containing background material from [5] on the definition of general almost modular functions and their limit theorems.

2. HOLOMORPHIC MODULAR FORMS

For any integer x and any prime p the standard quadratic residue symbol $\left(\frac{x}{p}\right)$ is 1 if x is a square modulo p , and -1 otherwise. The *generalized quadratic residue symbol* $\left(\frac{a}{b}\right)$ is, for any integer a and any odd integer b , characterized by the properties [4, pp. 160-161],

- (i) $\left(\frac{a}{b}\right) = 0$ if $(a, b) \neq 1$.
- (ii) $\left(\frac{a}{-1}\right) = \text{sgn } a$.
- (iii) If $b > 0$, $b = \prod_i b_i$, b_j primes, not necessarily distinct, then $\left(\frac{a}{b}\right) = \prod_i \left(\frac{a}{b_i}\right)$.
- (iv) $\left(\frac{a}{-b}\right) = \left(\frac{a}{-1}\right)\left(\frac{a}{b}\right)$.
- (v) $\left(\frac{0}{\pm 1}\right) = 1$.

It follows from these properties that the symbol is bimultiplicative:

$$(2.1) \quad \left(\frac{a_1 a_2}{b}\right) = \left(\frac{a_1}{b}\right) \left(\frac{a_2}{b}\right), \quad \left(\frac{a}{b_1 b_2}\right) = \left(\frac{a}{b_1}\right) \left(\frac{a}{b_2}\right).$$

Furthermore, if $b > 0$, then $\left(\frac{\cdot}{b}\right)$ defines a character modulo b ; if $a \neq 0$, then $\left(\frac{a}{\cdot}\right)$ defines a character modulo $4a$.

The action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})$ on the complex upper half plane $\mathfrak{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$ is defined by the fractional linear transformation $z \mapsto \frac{az+b}{cz+d}$. We are here interested in the congruence subgroups

$$(2.2) \quad \Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) : a \equiv d \equiv 1, c \equiv 0 \pmod{N} \right\}.$$

Fix $\Gamma = \Gamma_1(N)$, with $4|N$.

A *holomorphic modular form of weight κ for Γ* (with $\kappa \in \frac{1}{2}\mathbb{Z}$) is a holomorphic function $\mathfrak{H} \rightarrow \mathbb{C}$ that satisfies the functional relation

$$(2.3) \quad f(\gamma z) = \left(\frac{c}{d}\right)^{2\kappa} (cz + d)^\kappa f(z)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and that is holomorphic with respect to each cusp [6, Def. 1.3.3.]. This means that f has a Fourier expansion of the form

$$(2.4) \quad \sum_{m=0}^{\infty} \widehat{f}_m^{(i)} e(mz_i)$$

for each cuspidal coordinate z_i (cf. the appendix, Section 6); hence f is bounded in each cusp. In (2.3) the square root $z^{1/2}$ is chosen such that $-\pi/2 < \arg z^{1/2} \leq \pi/2$, and $z^{m/2} := (z^{1/2})^m$, for $m \in \mathbb{Z}$.

Famous examples of holomorphic modular forms are the theta series

$$(2.5) \quad \theta(z) = \sum_{n \in \mathbb{Z}} e(n^2 z),$$

which is of weight $\kappa = \frac{1}{2}$, and Jacobi's Δ -function

$$(2.6) \quad \Delta(z) = e(z) \prod_{n=1}^{\infty} (1 - e(nz))^{24},$$

where $\kappa = 12$, see [6].

Lemma 1. *Let $a_1, a_2, \dots, a_K \in \mathbb{C}$. Then the function*

$$(2.7) \quad \xi^{(K)}(z) = \sum_{k=1}^K a_k \theta(kz)$$

is a modular form of weight $\frac{1}{2}$ for $\Gamma_1(N)$ with $N = 4 \operatorname{lcm}(2, 3, \dots, K)$.

Proof. We have for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(4k)$

$$(2.8) \quad \theta \left(k \frac{az + b}{cz + d} \right) = \theta \left(\frac{a(kz) + kb}{(c/k)(kz) + d} \right) = \left(\frac{(c/k)}{d} \right) (cz + d)^{1/2} \theta(kz),$$

because

$$(2.9) \quad \begin{pmatrix} a & kb \\ c/k & d \end{pmatrix} \in \Gamma_1(4).$$

Since the generalized quadratic residue symbol is multiplicative,

$$(2.10) \quad \left(\frac{c}{d} \right) = \left(\frac{(c/k)}{d} \right) \left(\frac{k}{d} \right).$$

Furthermore $\left(\frac{k}{\cdot} \right)$ is a character mod $4k$, and hence, for $d \equiv 1 \pmod{4k}$, we have

$$(2.11) \quad \left(\frac{k}{d} \right) = \left(\frac{k}{1} \right) = 1.$$

This shows that $\theta^{(k)}(z) := \theta(kz)$ is a modular form for $\Gamma_1(4k)$. The lemma now follows from the observation that

$$(2.12) \quad \Gamma_1(N) \subset \bigcap_{k=1}^K \Gamma_1(4k).$$

□

3. HOLOMORPHIC ALMOST MODULAR FORMS

Definition 1. We call a holomorphic periodic function $\mathfrak{H} \rightarrow \mathbb{C}$

$$(3.1) \quad \xi(z) = \sum_{m=0}^{\infty} \widehat{\xi}_m e(mz)$$

a *holomorphic almost modular form (HAMF) of weight $\frac{1}{2}$* if for every $\epsilon > 0$ there is an $N = N(\epsilon)$ and a modular form

$$(3.2) \quad f_{\epsilon}(z) = \sum_{m=0}^{\infty} \widehat{f}_{\epsilon,m} e(mz)$$

of weight $\frac{1}{2}$ for $\Gamma_1(N)$, such that

$$(3.3) \quad \limsup_{M \rightarrow \infty} \frac{1}{\sqrt{M}} \sum_{m=0}^M |\widehat{\xi}_m - \widehat{f}_{\epsilon, m}|^2 < \epsilon^2.$$

To construct examples of such functions, let $h : \mathfrak{H} \rightarrow \mathbb{C}$ be a periodic holomorphic function of the form

$$(3.4) \quad h(z) = \sum_{k=1}^{\infty} \widehat{h}_k e(kz),$$

with constants $C > 0$ and $\beta > \frac{1}{4}$ such that for all $k \in \mathbb{N}$

$$(3.5) \quad |\widehat{h}_k| \leq \frac{C}{k^\beta}.$$

Theorem 2. *For h as in (3.4) and (3.5), the function*

$$(3.6) \quad \xi(z) = \sum_{n=1}^{\infty} h(n^2 z)$$

is a HAMF of weight $\frac{1}{2}$.

Proof. We choose as approximants the modular forms (cf. Lemma 1)

$$(3.7) \quad \xi^{(K)}(z) := \frac{1}{2} \sum_{k=1}^K \widehat{h}_k \theta(kz).$$

The Fourier coefficients of $\xi(z)$ are

$$(3.8) \quad \widehat{\xi}_0 = 0, \quad \widehat{\xi}_m = \sum_{k=1}^{\infty} \sum_{\substack{n=1 \\ m=n^2 k}}^{\infty} \widehat{h}_k,$$

and those of $\xi^{(K)}(z)$

$$(3.9) \quad \widehat{\xi}_0^{(K)} = \frac{1}{2} \sum_{k=1}^K \widehat{h}_k, \quad \widehat{\xi}_m^{(K)} = \sum_{k=1}^K \sum_{\substack{n=1 \\ m=n^2 k}}^{\infty} \widehat{h}_k.$$

Therefore

$$(3.10) \quad \begin{aligned} \sum_{m=1}^M \left| \widehat{\xi}_m - \widehat{\xi}_m^{(K)} \right|^2 &= \sum_{k_1, k_2=K+1}^{\infty} \sum_{\substack{n_1, n_2=1 \\ 1 \leq n_1^2 k_1 = n_2^2 k_2 \leq M}}^{\infty} \widehat{h}_{k_1} \overline{\widehat{h}_{k_2}} \\ &= \sum_{(p, q, r, s) \in S_1} \widehat{h}_{rp^2} \overline{\widehat{h}_{rq^2}} \leq C^2 \sum_{(p, q, r, s) \in S_1} \frac{1}{r^{2\beta} p^{2\beta} q^{2\beta}} \end{aligned}$$

where the sums are restricted to the set

$$(3.11) \quad S_1 = \{p, q, r, s \in \mathbb{N}, \quad \gcd(p, q) = 1, \quad rp^2, rq^2 > K, \quad 1 \leq rp^2 s^2 q^2 \leq M\}.$$

Thus

$$(3.12) \quad \sum_{m=1}^M \left| \widehat{\xi}_m - \widehat{\xi}_m^{(K)} \right|^2 \leq C^2 \sqrt{M} \sum_{(p,q,r) \in S_2} \frac{1}{r^{\frac{1}{2}+2\beta} p^{1+2\beta} q^{1+2\beta}} + O(1) \sum_{(p,q,r) \in S_3} \frac{1}{r^{2\beta} p^{2\beta} q^{2\beta}},$$

where

$$(3.13) \quad S_2 = \{p, q, r \in \mathbb{N}, \quad \gcd(p, q) = 1, \quad rp^2, rq^2 > K\},$$

$$(3.14) \quad S_3 = \{p, q, r \in \mathbb{N}, \quad \gcd(p, q) = 1, \quad 1 \leq rp^2 q^2 \leq M\}.$$

The last sum in (3.12) is bounded by (assume without loss of generality that $\frac{1}{4} < \beta < \frac{1}{2}$)

$$(3.15) \quad \sum_{(p,q,r) \in S_3} \frac{1}{r^{2\beta} p^{2\beta} q^{2\beta}} = O(M^{1-2\beta}) \sum_{\substack{p,q=1 \\ 1 \leq p^2 q^2 \leq M}}^{\infty} \frac{1}{p^{2-2\beta} q^{2-2\beta}} = O(M^{1-2\beta}).$$

Hence

$$(3.16) \quad \sum_{m=1}^M \left| \widehat{\xi}_m - \widehat{\xi}_m^{(K)} \right|^2 \leq C^2 \sqrt{M} \sum_{(p,q,r) \in S_2} \frac{1}{r^{\frac{1}{2}+2\beta} p^{1+2\beta} q^{1+2\beta}} + O(M^{1-2\beta}).$$

For $\beta > \frac{1}{4}$, the sum in (3.16) converges and tends to zero for K large. The remainder in (3.16) is of sub-leading order, i.e., $O(M^{1-2\beta}) = o(\sqrt{M})$. So, given any $\epsilon > 0$, there is a large K such that

$$(3.17) \quad \limsup_{M \rightarrow \infty} \frac{1}{\sqrt{M}} \sum_{m=0}^M \left| \widehat{\xi}_m - \widehat{\xi}_m^{(K)} \right|^2 < \epsilon^2.$$

□

If we choose $h(z) = \log(1 - e(z))$, we have $\widehat{h}_k = -1/k$ and thus Theorem 2 implies that

$$(3.18) \quad \xi(z) = \log \prod_{n=1}^{\infty} (1 - e(n^2 z))$$

is a HAMF.

4. THE LIMIT THEOREM

Theorem 3. *Let $\xi(z)$ be a HAMF of weight $\frac{1}{2}$. Then, for $x := \operatorname{Re} z$ uniformly distributed in $[0, 1)$ with respect to Lebesgue measure, $(\operatorname{Im} z)^{1/4} \xi(z)$ has a limit distribution as $y := \operatorname{Im} z \rightarrow 0$. That is, there exists a probability measure ν_ξ on \mathbb{C} such that, for any bounded continuous function $g : \mathbb{C} \rightarrow \mathbb{C}$, we have*

$$(4.1) \quad \lim_{y \rightarrow 0} \int_0^1 g(y^{1/4} \xi(x + iy)) dx = \int_{\mathbb{C}} g(w) \nu_\xi(dw).$$

Furthermore ν_ξ is invariant under rotations about the origin.

Proof. The following two lemmas show that Theorem 3 is a special case of the limit theorem for almost modular functions, Theorem 7. Rotational invariance of the limit distribution is proved at the end of this section. □

The manifold $\mathcal{M}_N = \Delta_1(N) \backslash \widetilde{\text{SL}}(2, \mathbb{R})$ and the function spaces $B_\sigma(\mathcal{M}_N)$, \mathcal{B}^2 , which appear below, are defined in the appendix (Section 6).

Lemma 4. *If $f(z)$ is a holomorphic modular form of weight κ for $\Gamma_1(N)$, then the function*

$$(4.2) \quad F(z, \phi) = (\text{Im } z)^{\kappa/2} f(z) e^{-i\kappa\phi}$$

is a modular function of class $B_{\kappa/2}(\mathcal{M}_N)$.

Proof. For $[\gamma, \beta_\gamma] \in \Delta_1(N)$ we have

$$(4.3) \quad F([\gamma, \beta_\gamma](z, \phi)) = [\text{Im}(\gamma z)]^{\kappa/2} f(\gamma z) e^{-i\kappa(\phi + \beta_\gamma)} = F(z, \phi)$$

because of (2.3) and

$$(4.4) \quad [\text{Im}(\gamma z)]^{\kappa/2} = \frac{(\text{Im } z)^{\kappa/2}}{|cz + d|^\kappa}, \quad e^{-i\kappa\beta_\gamma} = j_\gamma(z)^{-2\kappa} = \left\{ \left(\frac{c}{d} \right) \left(\frac{cz + d}{|cz + d|} \right)^{1/2} \right\}^{-2\kappa}.$$

Hence F is a smooth function on \mathcal{M}_N . Because (by definition) f is bounded in each cusp, we have

$$(4.5) \quad F(z, \phi) = O(y_i^{\kappa/2}),$$

cf. condition (6.6) in the appendix (Section 6). □

Lemma 5. *If $\xi(z)$ is a HAMF of weight $\frac{1}{2}$, then the function*

$$(4.6) \quad \Xi(z) = (\text{Im } z)^{1/4} \xi(z)$$

is an almost modular function of class \mathcal{B}^2 .

Proof. If f_ϵ are the approximants of ξ , we choose as approximants for Ξ the functions

$$(4.7) \quad F_\epsilon(z, \phi) = (\text{Im } z)^{1/4} f_\epsilon(z) e^{-i\phi/2}.$$

Then, in view of (3.3),

$$(4.8) \quad \limsup_{y \rightarrow 0} \int_0^1 |\Xi(x + iy) - F_\epsilon(x + iy, 0)|^2 dx = \limsup_{y \rightarrow 0} y^{1/2} \sum_{m=0}^{\infty} |\widehat{\xi}_m - \widehat{f}_{\epsilon, m}|^2 e^{-4\pi m y} = O(\epsilon^2),$$

and hence $\Xi \in \mathcal{B}^2$, see Definition 2. □

Proof of rotational invariance. The limit distribution for every approximant f_ϵ is given by (cf. Theorem 6)

$$(4.9) \quad \int_{\mathbb{C}} g(w) \nu_{f_\epsilon}(dw) = \int_{\mathcal{M}_N} g(y^{1/4} f_\epsilon(x + iy) e^{-i\phi/2}) \frac{dx dy d\phi}{y^2}.$$

Substituting $\phi + 2\omega$ for ϕ shows that $f_\epsilon(z)$ and $f_\epsilon(z) e^{-i\omega}$ have the same limit distribution for all $\omega \in [0, 2\pi)$. Hence $\xi(z)$ and $\xi(z) e^{-i\omega}$ share the same limit distribution. □

5. THE VARIANCE

Let us return to the example introduced in Theorem 2 and derive an explicit formula for the variance of the limit distribution. By slightly modifying the steps in (3.10), (3.12) one finds that for

$$(5.1) \quad \alpha(t) := \sum_{0 \leq m < t} |\widehat{\xi}_m|^2$$

and $t \rightarrow \infty$ we have

$$(5.2) \quad \alpha(t) \sim A t^{1/2}, \quad \text{with } A := \sum_{r=1}^{\infty} \sum_{\substack{p,q=1 \\ \gcd(p,q)=1}}^{\infty} \frac{\widehat{h}_{rp^2} \overline{\widehat{h}_{rq^2}}}{pq\sqrt{r}}.$$

Therefore, using Parseval's equality and a standard Abelian Theorem for the Laplace transform [7, Chap. V],

$$(5.3) \quad \int_0^1 |\xi(x + iy)|^2 dx = \sum_{m=0}^{\infty} |\widehat{\xi}_m|^2 e^{-4\pi my} = \int_0^{\infty} e^{-4\pi yt} d\alpha(t) \sim A \Gamma(\tfrac{3}{2}) (4\pi y)^{-1/2},$$

as $y \rightarrow 0$. Since Euler's function evaluates to $\sqrt{\pi}/2$, we have

$$(5.4) \quad \lim_{y \rightarrow 0} y^{1/2} \int_0^1 |\xi(x + iy)|^2 dx = \frac{1}{4} \sum_{r=1}^{\infty} \sum_{\substack{p,q=1 \\ \gcd(p,q)=1}}^{\infty} \frac{\widehat{h}_{rp^2} \overline{\widehat{h}_{rq^2}}}{pq\sqrt{r}}.$$

6. APPENDIX: ALMOST MODULAR FUNCTIONS FOR $\widetilde{\text{SL}}(2, \mathbb{R})$

This appendix provides some background material on the definition of almost modular functions and their limit theorems; the reader is referred to [5] for more detailed information.

Let us denote by $C(\mathfrak{H})$ the space of continuous functions $\mathfrak{H} \rightarrow \mathbb{C}$, and put $\epsilon_g(z) = (cz + d)/|cz + d|$. The universal covering group of $\text{SL}(2, \mathbb{R})$ is defined as the set

$$(6.1) \quad \widetilde{\text{SL}}(2, \mathbb{R}) = \{[g, \beta_g] : g \in \text{SL}(2, \mathbb{R}), \beta_g \in C(\mathfrak{H}) \text{ such that } e^{i\beta_g(z)} = \epsilon_g(z)\},$$

with multiplication law

$$(6.2) \quad [g, \beta_g^1][h, \beta_h^2] = [gh, \beta_{gh}^3], \quad \beta_{gh}^3(z) = \beta_g^1(hz) + \beta_h^2(z).$$

We may identify $\widetilde{\text{SL}}(2, \mathbb{R})$ with $\mathfrak{H} \times \mathbb{R}$ via $[g, \beta_g] \mapsto (z, \phi) = (g\mathbf{i}, \beta_g(\mathbf{i}))$. The action of $\widetilde{\text{SL}}(2, \mathbb{R})$ on $\mathfrak{H} \times \mathbb{R}$ is then $[g, \beta_g](z, \phi) = (gz, \phi + \beta_g(z))$. The Haar measure of $\widetilde{\text{SL}}(2, \mathbb{R})$ reads, in this parametrization,

$$(6.3) \quad d\mu(g) = \frac{dx dy d\phi}{y^2}.$$

The group $\Delta_1(N)$ is the following discrete subgroup of $\widetilde{\text{SL}}(2, \mathbb{R})$,

$$(6.4) \quad \Delta_1(N) = \{[\gamma, \beta_\gamma] : \gamma \in \Gamma_1(N), \beta_\gamma \in C(\mathfrak{H}) \text{ such that } e^{i\beta_\gamma(z)/2} = j_\gamma(z)\},$$

where

$$(6.5) \quad j_\gamma(z) = \left(\frac{c}{d}\right) \left(\frac{cz+d}{|cz+d|}\right)^{1/2}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(4).$$

Here $z^{1/2}$ denotes the principal branch of the square-root of z , i.e., the one for which $-\pi/2 < \arg z^{1/2} \leq \pi/2$.

The homogeneous space $\mathcal{M}_N = \Delta_1(N) \backslash \widetilde{\text{SL}}(2, \mathbb{R})$ has finite volume with respect to Haar measure (6.3). \mathcal{M}_N has a finite number of cusps, which are represented by the set $\eta_1, \dots, \eta_\kappa \in \mathbb{Q} \cup \infty$ on the boundary of \mathfrak{H} . Let γ_i be a fractional linear transformation $\mathfrak{H} \rightarrow \mathfrak{H}$ which maps the cusp at η_i to the standard cusp at ∞ of width one. Thus $(z_i, \phi_i) = \tilde{\gamma}_i(z, \phi)$ yields a new set of coordinates, where the i th cusp appears as a cusp at ∞ , which is invariant under $(z_i, \phi_i) \mapsto (z_i + 1, \phi_i)$. The variable $y_i = \text{Im}(\gamma_i z)$ measures the height into the i th cusp. For any $\sigma \geq 0$, we denote by $B_\sigma(\mathcal{M}_N)$ the class of functions $F \in C(\mathcal{M}_N)$ such that, for all $i = 1, \dots, \kappa$,

$$(6.6) \quad F(z, \phi) = O(y_i^\sigma), \quad y_i \rightarrow \infty,$$

where the implied constant is independent of (z, ϕ) . In view of the form of the invariant measure (6.3) we note that $B_\sigma(\mathcal{M}_N) \subset L^p(\mathcal{M}_N, \mu)$ if $\sigma < 1/p$.

The following theorem [5, Th. 6.1] states that the closed horocycles

$$(6.7) \quad \Delta_1(N) \{(x + iy, 0) : x \in [0, 1)\}$$

are asymptotically equidistributed in \mathcal{M}_N , as $y \rightarrow 0$.

Theorem 6. *Let $0 \leq \sigma < 1$. Then, for every $F \in B_\sigma(\mathcal{M}_N)$, we have*

$$(6.8) \quad \lim_{y \rightarrow 0} \int_0^1 F(x + iy, 0) dx = \frac{1}{\mu(\mathcal{M}_N)} \int_{\mathcal{M}_N} F d\mu.$$

Let us now turn to the definition of *almost modular functions of class \mathcal{B}^p or \mathcal{H}* , respectively, as given in [5, Sect. 7]. In the following we will consider functions $\Xi : \mathfrak{H} \rightarrow \mathbb{C}$, which are *periodic*, i.e., for which $\Xi(z + 1) = \Xi(z)$.

Definition 2. For any $p \geq 1$, let \mathcal{B}^p be the class of periodic functions $\Xi : \mathfrak{H} \rightarrow \mathbb{C}$ with the property that for every $\epsilon > 0$ there are an integer $N = N(\epsilon) > 0$ and a function $F_\epsilon \in B_\sigma(\mathcal{M}_N)$ with $0 \leq \sigma < 1/p$ so that

$$(6.9) \quad \limsup_{y \rightarrow 0} \int_0^1 |\Xi(x + iy) - F_\epsilon(x + iy, 0)|^p dx < \epsilon^p.$$

Definition 3. Let \mathcal{H} be the class of periodic functions $\Xi : \mathfrak{H} \rightarrow \mathbb{C}$ with the property that for every $\epsilon > 0$ there are an integer $N = N(\epsilon) > 0$ and a bounded continuous function $F_\epsilon \in C(\mathcal{M}_N)$ such that

$$(6.10) \quad \limsup_{y \rightarrow 0} \int_0^1 \min \{1, |\Xi(x + iy) - F_\epsilon(x + iy, 0)|\} dx < \epsilon.$$

If $1 \leq q \leq p$ we have the inclusion $\mathcal{B}^p \subset \mathcal{B}^q \subset \mathcal{H}$, see [5, Prop. 7.3]. The central observation of [5] is the following limit theorem for almost modular functions [5, Th. 8.2].

Theorem 7. *Let $\Xi \in \mathcal{H}$. Then, for x uniformly distributed in $[0, 1)$ with respect to Lebesgue measure, $\Xi(x + iy)$ has a limit distribution as $y \rightarrow 0$. That is, there exists a probability measure ν_Ξ on \mathbb{C} such that, for every bounded continuous function $g : \mathbb{C} \rightarrow \mathbb{C}$,*

$$(6.11) \quad \lim_{y \rightarrow 0} \int_0^1 g(\Xi(x + iy)) dx = \int_{\mathbb{C}} g(w) \nu_\Xi(dw).$$

The proof of this theorem follows closely the argument for almost periodic functions [1]. The main difference is that the equidistribution theorem for irrational Kronecker flows on multidimensional tori is here replaced by Theorem 6, cf. [5, Sect. 8].

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